Supersonic Flows - Non Linear Analysis

* Supersonic Flows may not be smooth (for example shock waves may occur with jump in thermodynamic variables)

* Even for small disturbances there are jumps across Mach lines which can be considered as weak shocks.

* The derivative of a step function is a delta function.

* In Inviscid flow shocks are discontinuous but still mass, momentum & Energy are conserved across shocks.

Approach I:

use differential equation for smooth flow upstream & downstream of the shock & use the conservation laws across the shock.

Approach II:

use conservation laws everywhere "integral form rather than differential form"
Normal Shock Waves:

"Quasi One Dim flow in Nozzle"

Mass: \( \rho_1 u_1 \rho_1 \approx \rho_2 u_2 \rho_2 \approx m \)

Momentum: \( p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 = H \)

Energy: \( \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} + \frac{1}{2} u_1^2 = \frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2} + \frac{1}{2} u_2^2 = H \)

where \( H = p + \frac{1}{2} u^2 \), \( H = C_P T = \frac{C_P}{\gamma} \frac{p}{\rho} = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \)

* Conservation laws are 3 Eq in 3 unknowns \( \rho, u, p \) although of their nonlinearity they can be solved using elimination.

\[
p = M - mu \Rightarrow H = \frac{\gamma}{\gamma - 1} \frac{M - mu}{\rho} + \frac{1}{2} u^2
\]

\[
H = \frac{\gamma}{\gamma - 1} \frac{(M - mu) u}{m} + \frac{1}{2} u^2 \quad \text{"Quadratic Eq in } u\" 
\]

Two solutions exist \( u'_2 \) & \( u''_2 \) such that

\[
u'_2 u''_2 = a^2 \Rightarrow \frac{a^2}{\gamma - 1} + \frac{1}{2} a^2 = H
\]

\[
a^2 = \frac{2(\gamma - 1)}{\gamma + 1} H \quad \text{"Critical Speed of Sound"}
\]
Note that \( u_t = \frac{\sqrt{b^2 - 4ac}}{2a} \) \( \Rightarrow u' u'' = \frac{b^2 - b^2 + 4ac}{a^2} = \frac{c}{a} \)

\( c = -H \) \( \Rightarrow \frac{a}{b - 1} \Rightarrow u' u'' = H \left[ \frac{b + 1}{2(b - 1)} \right]^{-1} = \frac{a}{b} \)

Now define \( M^* = \frac{u}{\sigma^*} \) \( \Rightarrow M^*_1 M^*_2 = 1 \) Prandtl Relation

\( \Rightarrow 3 \) Possibilities:

\( (a) \) \( M^*_1 = M^*_2 = 1 \) "Sonic Condition"

\( (b) \) \( M^*_1 < 1 \) \( M^*_2 > 1 \) \( \Rightarrow \Delta S < 0 \)

& this contradicts the 2nd law of Thermo.

\( (c) \) \( M^*_1 > 1 \) \( M^*_2 < 1 \) "Super to Subsonic Shock"

\( \Rightarrow \Delta S > 0 \)

# Relation between \( M \) & \( M^* \):

\( H = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} + \frac{1}{2} u^2 = \frac{\sigma^*}{b - 1} + \frac{1}{2} u^2 \)

\( = \frac{\sigma^*}{b - 1} + \frac{1}{2} \sigma^* \)

\( \Rightarrow \frac{1}{\gamma - 1} \frac{1}{M^2} + \frac{1}{2} \frac{1}{M^*} = \frac{1}{\gamma - 1} \frac{1}{M^*} + \frac{1}{2} \frac{1}{M^*} \)
Solving for $M^{*2}$

\[
M^{*2} = \frac{2}{\frac{M^2}{M^{*2}} + (\sigma - 1)}
\]

Solving for $M^2$

\[
M^2 = \frac{2}{\frac{(\sigma + 1)}{M^{*2}} - (\sigma - 1)}
\]

Note that:
- $M = 1 \Rightarrow M^* = 1$
- $M < 1 \Rightarrow M^* < 1$
- $M > 1 \Rightarrow M^* > 1$
- $M \rightarrow \infty \Rightarrow M^* = \frac{2}{\sigma - 1} = 6 \text{ "Div"}$

⇒ Normal Shock jump Condition in terms of $M_1^*$:

\[
\frac{P_2}{P_1} = \frac{u_2}{u_1} = M_1^{*2} = \frac{(\sigma + 1) M_1^2}{2 + (\sigma - 1) M_1^2}
\]

Also

\[
P_2 - P_1 = \rho u_1^2 - \rho u_1 u_2 = \rho u_1^2 \left( 1 - \frac{u_2}{u_1} \right)
\]

⇒

\[
\frac{P_2 - P_1}{P_1} = \frac{\rho u_1^2}{P_1} \left( 1 - \frac{u_2}{u_1} \right) \Rightarrow \frac{P_2}{P_1} = 1 + \frac{2(\sigma - 1)}{\sigma + 1} (M_1^{*2} - 1)
\]

Also

\[
\frac{T_2}{T_1} = \frac{\sigma_2}{\sigma_1} = 1 + \frac{2(\sigma - 1)}{(\sigma + 1)^2} \frac{\sigma M_1^2 + 1}{M_1^2} \left( M_1^{*2} - 1 \right)
\]

\[
\frac{S_2 - S_1}{R} = \ln \left[ \frac{\rho_2}{\rho_1} \left( \frac{\sigma_2}{\sigma_1} \right)^{\frac{\gamma}{2}} \right]
\]
# Oblique Shocks:

\[ u_1 = W_1 \sin \beta, \quad u_2 = W_2 \sin (\beta - \theta) \]

\[ H_1 = \frac{W_1}{\theta_1}, \quad H_2 = \frac{W_2}{\theta_2} \]

\[ \frac{u_1}{\theta_1} = \frac{W_1 \sin \beta}{\theta_1} = H_1 \sin \beta \]

Tangential Comp. is conserved: \( u_1 = u_2 \)

Normal Shock Relations are valid using the velocity components normal to the shock:

\[ \frac{p_2}{p_1} = \frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) H_1^2 \sin^2 \beta}{(\gamma - 1) H_1^2 \sin^2 \beta + 2} \]

\[ \ldots \text{and so on} \]

# Relation between \( \beta \) & \( \theta \):

\[ \tan (\beta - \theta) = \frac{u_2}{u_1} \]

Shock due to Compression

# Prandtl Relation for Oblique Shock:

\[ u_1 u_2 = \gamma \frac{p_2}{p_1} - \frac{\gamma - 1}{\gamma + 1} \frac{v^2}{2} \]
Prandtl-Meyer Expansion:

- \( \theta = \sqrt{M^2-1} \frac{SW}{w} \)

\[ \Rightarrow \theta + \text{Const} = \int \sqrt{M^2-1} \frac{dw}{w} = \psi(M) \]

but \( W = \rho M \Rightarrow \frac{5W}{w} = \frac{\theta}{W} \rho M + M \frac{5\theta}{M} \frac{\rho}{\theta} \)

\[ H = \frac{\rho^2}{\theta^2} + \frac{1}{2} \frac{W^2}{\theta^2} \Rightarrow \frac{2 \frac{\delta \theta}{\delta M}}{\theta-1} + W \delta W = 0 \]

\[ \Rightarrow \frac{\delta \theta}{\delta M} = -\frac{(\theta-1)}{2} \frac{\delta W}{W} \]

\[ \Rightarrow \frac{\delta W}{\delta M} = \frac{5M}{M} - \frac{\theta-1}{2} \frac{H^2}{W} \Rightarrow \frac{SW}{W} = \frac{1}{2} \frac{H^2}{M} \frac{5W}{W} \]

\[ \Rightarrow U = \theta + C = \int \sqrt{\frac{M^2-1}{1 + \frac{\delta \theta}{\delta M} \frac{H^2}{W}}} \frac{dH}{M} \]

\[ = \sqrt{\frac{\theta+1}{\theta-1}} \tan^{-1} \left[ \frac{\theta-1}{\theta+1} \frac{(H^2-1)}{H^2} \right] - \tan^{-2} \frac{H^2-1}{H^2} \]

\[ \Delta U = 2 \theta(H_1) - 2 \theta(H_2) \quad 130.45^\circ \]

1 \rightarrow \rightarrow H
# Conservation Form:

**One Dim**:  
Mass: \((\rho u A)_x = 0\)

Momentum: \((A\rho u)^x + (AP)_x = A_x P\)

Energy: \((\rho u A H)_x = 0\)

**Two Dim**:  
\((\rho u)_x + (\rho v)_y = 0\)

\((\rho u^2)_x + (\rho uv)_y = -P_x\)

\((\rho uv)_x + (\rho v^2)_y = -P_y\)

\((\rho u H)_x + (\rho u H)_y = 0\)

\[H = \frac{\rho}{\sigma - 1} + \frac{1}{2} (u^2 + v^2)\]

Applying Gauss Divergence Theorem

\[\iint (f_x + g_y) \, dx \, dy = \oint f \, dx - g \, dx\]

No derivatives

"Valid across shocks"