Fourier Series

If $f(x)$ is a periodic function of period 2π, it can be represented by a trignometric series

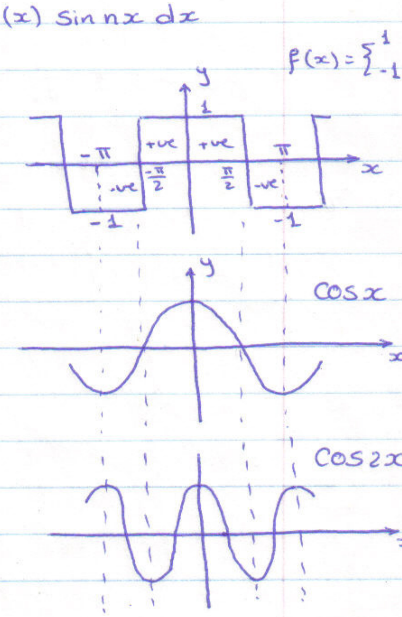
$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$



Example 1 :-

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{-\pi/2} -\cos nx \, dx$$

$$+ \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos nx \, dx$$

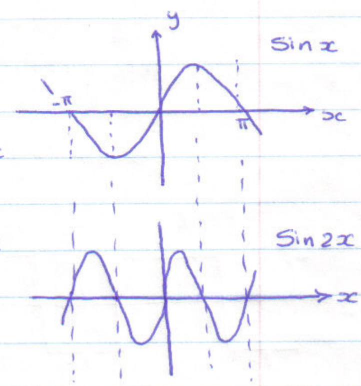
$$+ \frac{1}{\pi} \int_{\pi/2}^{\pi} -\cos nx \, dx$$

$$a_n = \frac{1}{\pi}\left[\frac{-\sin nx}{n}\Big|_{-\pi}^{-\pi/2} + \frac{\sin nx}{n}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{-\sin nx}{n}\Big|_{\frac{\pi}{2}}^{\pi} \right]$$

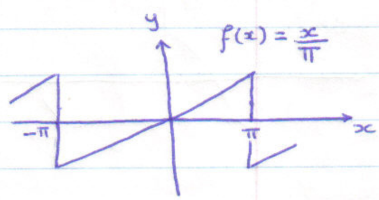
$$= \frac{1}{\pi}\left[\pm\frac{1}{n} \pm \frac{2}{n} \pm \frac{1}{n} \right] = \pm\frac{4}{n\pi}$$

$$\therefore \quad a_n = \begin{cases} \frac{4}{n\pi} & n = 1, 5, 9, \ldots \\ -\frac{4}{n\pi} & n = 3, 7, 11, \ldots \\ \text{zero} & n = 2, 4, 6, \ldots \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin nx}_{\text{odd function}}\, dx = 0$$



$$\therefore \; f(x) = \frac{4}{\pi}\cos x - \frac{4}{3\pi}\cos 3x + \frac{4}{5\pi}\cos 5x - \frac{4}{7\pi}\cos 7x + \ldots$$



Example 2:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\cos nx}_{\text{odd function}}\, dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{x}{\pi} \sin nx \, dx$$

$$= \frac{-1}{\pi^2} \int_{-\pi}^{\pi} \frac{x}{n} \, d \cos(nx)$$

$$= -\frac{1}{\pi^2} \left[\frac{x}{n} \cos nx \Big|_{-\pi}^{\pi} - \frac{1}{n} \int_{-\pi}^{\pi} \cos nx \, dx \right] \quad (\text{integral} = 0)$$

$$= -\frac{1}{\pi^2} \left[\frac{\pi}{n} (\mp 1) + \frac{\pi}{n} (\mp 1) \right] = \pm \frac{2}{n\pi}$$

$$\therefore \quad b_n = \begin{cases} \frac{2}{n\pi} & n = 1, 3, 5, \ldots \\ \frac{-2}{n\pi} & n = 2, 4, 6, \ldots \end{cases}$$

$$\Rightarrow \quad f(x) = \frac{2}{\pi} \sin x - \frac{2}{2\pi} \sin 2x + \frac{2}{3\pi} \sin 3x - \ldots$$