# Stability Analysis

let

$A =$ Exact Analytical Solution

$E =$ Exact Solution for discretized Equation

$N =$ Computer Solution for discretized Equation

$\Rightarrow$ $A - D =$ truncation Error

$\Rightarrow$ $N - D =$ round-off Error $= \epsilon$

$$N = D + \varepsilon$$

** Studying the Equation $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$

$$\Rightarrow \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \frac{u_{i+1}^n - 2 u_i^n + u_{i-1}^n}{(\Delta y)^2} + O\left[\Delta t, \Delta y^2\right]$$

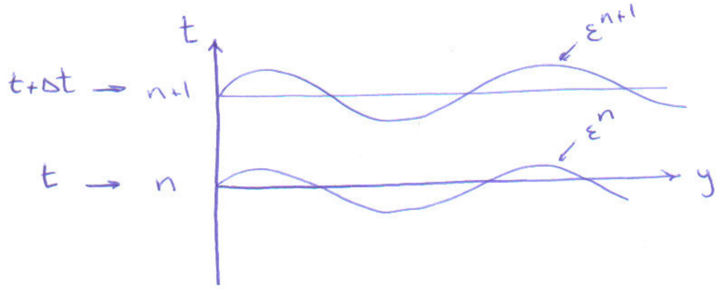
$$\Rightarrow \quad \frac{(\emptyset + \varepsilon)_i^{n+1} - (\emptyset + \varepsilon)_i^n}{\Delta t} = \nu \frac{(\emptyset + \varepsilon)_{i+1}^n - 2(\emptyset + \varepsilon)_i^n + (\emptyset + \varepsilon)_{i-1}^n}{(\Delta y)^2}$$

$$\Rightarrow \quad \frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \nu \frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{(\Delta y)^2}$$

∴ Round off Error is a function of t & y

*** <u>Von Neumann Stability Analysis</u>:-

$$\varepsilon(t,y) = e^{at}\, e^{iK_m y}$$




$$\therefore \quad e^{a(t+\Delta t)}\, e^{iK_m y} - e^{at}\, e^{iK_m y} = \frac{\nu \Delta t}{\Delta y^2} * e^{at}\left[ e^{iK_m(y+\Delta y)} - 2e^{iK_m y} + e^{iK_m(y-\Delta y)} \right]$$

let $r = \frac{\nu \Delta t}{\Delta y^2}$ , $\beta = K_m \Delta y$

$$\Rightarrow \quad e^{a\Delta t} - 1 = r\left[ e^{iK_m\Delta y} - 2 + e^{-iK_m \Delta y} \right]$$

$$\Rightarrow \quad e^{a\Delta t} - 1 = r\left[ e^{i\beta} + e^{-i\beta} - 2 \right]$$

$$= r\left[ \cos\beta + i\cancel{\sin\beta} + \cos\beta - i\cancel{\sin\beta} - 2 \right]$$

$$= 2r\left[ \cos\beta - 1 \right]$$

$$= 2r\left[ -2\sin^2\frac{\beta}{2} \right] = -4r\sin^2\frac{\beta}{2}$$

$\therefore \quad e^{a\Delta t} = 1 - 4r\sin^2\frac{\beta}{2}$

$\Rightarrow \quad \varepsilon_i^n = e^{at}e^{i\beta} \quad , \quad \varepsilon_i^{n+1} = e^{a(t+\Delta t)}e^{i\beta}$

$\therefore$ Amplification factor $= \dfrac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = e^{a\Delta t}$ $\begin{cases} >1 & \text{Unstable} \\ <1 & \text{Stable} \end{cases}$

$$\therefore \quad \left| 1 - 4r\sin^2\frac{\beta}{2} \right| \leqslant 1$$

<u>Stability Condition</u> :-

$$r \geqslant 0 \qquad \Rightarrow \qquad \frac{\nu\,\Delta t}{\Delta y^2} \geqslant 0$$

$$\& \qquad r \leqslant \frac{1}{2\sin^2\beta/2} \quad \Rightarrow \quad \Delta t \leqslant \frac{1}{2}\frac{\Delta y^2}{\nu}$$

$\therefore$ for stability use smaller $\Delta t$

<u>N.B.</u>

Explicit methods only suffer's stability problems.

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