Universal Laws:

1. Conservation of Mass
   "Continuity Equation"

2. Conservation of Momentum
   "Momentum Equation"
   "Newton's 2nd law"

3. Conservation of Energy
   "Energy Equation"
   "1st law of Thermodynamics"

A. Continuity Equation:

\[ \rho \frac{d}{dx} \left( \frac{\rho u^2}{2} \right) + \rho \frac{d}{dy} \left( \rho u v \right) + \rho \frac{d}{dz} \left( \rho u w \right) = \left[ \rho u + \frac{\partial (\rho u^2)}{\partial y} \right] \right|_{x=1}^{x=0} + \left[ \rho u + \frac{\partial (\rho u v)}{\partial z} \right] \right|_{y=1}^{y=0} \]

Right-Hand Rule is to be applied

"Infinitesimal Control volume"

(1/4)
\[ \frac{\partial}{\partial t} \left( \rho \, dx \, dy \, dz \right) = - \left( \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} \right) dxdydz \]

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = 0 \]

\[ \vec{V} = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \]

\[ \nabla p = \frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} + \frac{\partial p}{\partial z} \hat{k} \]

\[ \nabla \cdot \vec{q} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial q}{\partial z} \]

\[ \nabla \times \vec{q} = \left( \frac{\partial q}{\partial y} - \frac{\partial q}{\partial z} \right) \hat{i} - \left( \frac{\partial q}{\partial x} - \frac{\partial q}{\partial z} \right) \hat{j} + \left( \frac{\partial q}{\partial x} - \frac{\partial q}{\partial y} \right) \hat{k} \]

\[ \text{Curl of } \vec{q} \Rightarrow \text{indicates fluid vorticity} \]

\[ (21) \]
B. Momentum Equation:

\[ \sum \vec{F} = \frac{D}{Dt} \quad \text{(Momentum)} \]

\[ \Rightarrow \sum \vec{F} = \frac{D}{Dt} (m \vec{\dot{q}}) = m * \frac{D \vec{q}}{Dt} \]

where \( \frac{D}{Dt} = \text{Material/Substantial Derivative}. \)

Note:

\[ \frac{dc}{dt} = \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} + w \frac{\partial c}{\partial z} \]

variation of \( c \) along a streamline

\[ \frac{Dc}{Dt} = \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} + w \frac{\partial c}{\partial z} \]

\( u = \text{velocity of fluid in } x \)-direction

\( (3/11) \)
\[ \sum \mathbf{F} = m \frac{D\mathbf{q}}{Dt} = m \left[ \frac{\partial \mathbf{q}}{\partial t} + \mathbf{u} \frac{\partial \mathbf{q}}{\partial x} + \mathbf{v} \frac{\partial \mathbf{q}}{\partial y} + \mathbf{w} \frac{\partial \mathbf{q}}{\partial z} \right] \]

**Temporal** term
**Convective term**
"Non linear"

\[ F_x = m \left[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \frac{\partial \mathbf{u}}{\partial x} + \mathbf{v} \frac{\partial \mathbf{u}}{\partial y} + \mathbf{w} \frac{\partial \mathbf{u}}{\partial z} \right] \]
\[ F_y = m \left[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{u} \frac{\partial \mathbf{v}}{\partial x} + \mathbf{v} \frac{\partial \mathbf{v}}{\partial y} + \mathbf{w} \frac{\partial \mathbf{v}}{\partial z} \right] \]
\[ F_z = m \left[ \frac{\partial \mathbf{w}}{\partial t} + \mathbf{u} \frac{\partial \mathbf{w}}{\partial x} + \mathbf{v} \frac{\partial \mathbf{w}}{\partial y} + \mathbf{w} \frac{\partial \mathbf{w}}{\partial z} \right] \]

Forces acting on fluid

**Body Forces**
* Weight
* Electromagnetic

**Surface**
* Pressure force
* Viscous Stresses

\[ \int (p + \frac{\partial \rho}{\partial y} dy) \, dx \, dz \]
\[ p \, dx \, dy \]
\[ \int [p + \frac{\partial p}{\partial z} dx] \, dy \, dz \]
\[ (p + \frac{\partial p}{\partial z}) \, dx \, dy \]
\[ p \, dx \, dz \]

(4/11)
Pressure force in $x$-direction = $-\frac{\partial P}{\partial x} \, dx \, dy \, dz$

Pressure force in $y$-direction = $-\frac{\partial P}{\partial y} \, dx \, dy \, dz$

Pressure force in $z$-direction = $-\frac{\partial P}{\partial z} \, dx \, dy \, dz$

$\Rightarrow$ Stress tensor $\tau_{ij}$:

\[
\tau_{ij} = \begin{bmatrix}
\tau_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \tau_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \tau_{zz}
\end{bmatrix}
\]

2nd order tensor has 9 components

$i \Rightarrow$ plane $\perp$ axis, $j \Rightarrow$ direction

\[\tau_{yx} \, dx \, dy \, dz \quad \tau_{zy} \, dy \, dz \quad \tau_{zx} \, dx \, dz \quad \tau_{yz} \, dx \, dy \quad \tau_{xy} \, dy \, dz \quad \tau_{xz} \, dz \, dx \quad \tau_{yz} \, dz \, dy \quad \tau_{zx} \, dy \, dx \quad \tau_{yz} \, dx \, dz \quad \tau_{zy} \, dy \, dx \quad \tau_{yx} \, dz \, dy \]

$\Rightarrow$ (5/11)
Viscous force in $x$-direction

$$= \left[ \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right] dx \, dy \, dz$$

Body force in $x$-direction

$$= \rho f_x \, dx \, dy \, dz$$

**X-Momentum Equation**

$$\rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right]$$

$$= \rho f_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

**General form of Momentum Equation**

$$\rho \left[ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right] = \rho \vec{f} - \nabla p + \nabla \cdot \vec{\tau}$$

where

$$\vec{q} \cdot \vec{v} = \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right)$$

(6/11)
Newtonian Fluid

"Linear relation between stress & rate of strain"

\[ \tau_{ij} = \mu \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} \]

\[ \tau_{xx} = 2 \mu \frac{\partial u}{\partial x} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \]

\[ \tau_{yy} = 2 \mu \frac{\partial v}{\partial y} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \]

\[ \tau_{zz} = 2 \mu \frac{\partial w}{\partial z} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \]

\[ \tau_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \tau_{yx} \]

\[ \tau_{xz} = \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \tau_{zx} \]

\[ \tau_{yz} = \mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \tau_{zy} \]

where

\[ \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \]

\[ \mu = \text{viscosity coeff.} \]

\[ \lambda = \text{second viscosity coeff.} \]

(7/11)
# Stokes Hypothesis

\[ \tau_{xx} + \tau_{yy} + \tau_{zz} = 0 \]

\[ \Rightarrow 2\mu \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] + 3\lambda \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] = 0 \]

\[ \Rightarrow 2\mu + 3\lambda = 0 \quad \Rightarrow \quad \lambda = -\frac{2}{3}\mu \]

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C. Energy Equation

\[ SQ + SW = \frac{DE}{dt} \quad \text{"1st law of Thermodynamics applied on Control Mass"} \]

where;  \( SQ = \) Heat transferred to system  
\( SW = \) Work done on system

Remember that

\[ \frac{D}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \]

# Sources of Heat

- Volumetric Heating
- Heat Conduction

(8111)
The relaxation technique is an iterative method wherein values of four of the quantities are assumed to be the known values at iteration step \( n \) & \( i \):

\[
\phi_{i,j}^{n+1} = \frac{(\Delta x)^2 (\Delta y)^2}{2 (\Delta x)^2 + 2 (\Delta x)^2} \left[ \frac{\phi_{i+1,j}^{n} + \phi_{i-1,j}^{n}}{(\Delta x)^2} + \frac{\phi_{i,j+1}^{n} + \phi_{i,j-1}^{n}}{(\Delta y)^2} \right]
\]

1) We first assume values for \( \phi \) at all grid points except one, at which \( \phi \) is treated as the unknown.

2) Then the equation is applied at all internal grid points & first iteration \( n=1 \) is finished & we go on to the next step.

It is suggested that updated values of \( \phi \) be used as soon as possible on the right-hand side of the equation.

Convergence is achieved when \( \phi_{i,j}^{n+1} - \phi_{i,j}^{n} \) becomes less than some prescribed value at all grid points.

Frequently, the convergence to a solution sometimes can be enhanced by a technique called successive overrelaxation \( (\omega>1) \):

\[
\phi_{i,j}^{n+1} = \phi_{i,j}^{n} + \omega \left( \overline{\phi}_{i,j}^{n+1} - \phi_{i,j}^{n} \right)
\]

(9110)
# Work done on System:

**Notation:**

1. If force & velocity are in same direction then work is done on the element.

2. If force & velocity are in opposite directions then work is done by the system.

\[
\begin{align*}
\int_{\Omega} \left[ z_{yy} u + \frac{\partial (z_{yy} u)}{\partial y} \right] \, dx \, dz \\
\int_{\Omega} \left[ \rho u + \frac{\partial (\rho u)}{\partial x} \right] \, dy \, dz \\
\int_{\Omega} \left[ z_{xx} u + \frac{\partial (z_{xx} u)}{\partial x} \right] \, dy \, dz \\
\int_{\Omega} z_{xx} u \, dx \, dz
\end{align*}
\]

So Net Work done on the System by x-direction forces:

\[
\begin{align*}
\int_{\Omega} \left[ \frac{\partial \rho u}{\partial x} + \frac{\partial (z_{xx} u)}{\partial x} \right] \, dx \, dy \, dz \\
+ \int_{\Omega} \left[ \frac{\partial (z_{yy} u)}{\partial y} + \frac{\partial (z_{xx} u)}{\partial z} \right] \, dx \, dy \, dz
\end{align*}
\]

\[10/11\]
# Energy Equation:

\[
\rho \frac{D\mathcal{E}}{Dt} = \rho \mathbf{q} \cdot \mathbf{i} + \rho \left[ \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) \right] \\
- \frac{\partial (p\mathbf{u})}{\partial x} - \frac{\partial (p\mathbf{v})}{\partial y} - \frac{\partial (p\mathbf{w})}{\partial z} + \frac{\partial (c_{xx} \mathbf{u})}{\partial x} + \\
\frac{\partial (c_{yy} \mathbf{u})}{\partial y} + \frac{\partial (c_{zz} \mathbf{u})}{\partial z} + \frac{\partial (c_{xy} \mathbf{u})}{\partial x} + \frac{\partial (c_{yz} \mathbf{u})}{\partial y} + \frac{\partial (c_{xz} \mathbf{u})}{\partial z} \]

OR

\[
\rho \frac{D\mathcal{E}}{Dt} = \rho \mathbf{q} \cdot \mathbf{i} + \nabla \cdot (k \nabla T) - \nabla \cdot (\rho \mathbf{q}) + \nabla \cdot \mathbf{\kappa} \mathbf{q} \]

Notes

(i) for incompressible fluids, energy equation can be decoupled & used to obtain temperature only

(ii) further simplification can be introduced by considering incompressible & inviscid flows.