The Modified Equation

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0 \quad , \quad c > 0$$

$$\Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0 \longrightarrow (I)$$

T.S.E.

$$u_i^{n+1} = u_i^n + \Delta t\, u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} + \cdots \longrightarrow (II)$$

$$u_{i-1}^n = u_i^n - \Delta x\, u_x + \frac{\Delta x^2}{2} u_{xx} - \frac{\Delta x^3}{6} u_{xxx} + \cdots \longrightarrow (III)$$

from I :- $\quad u_i^{n+1} - u_i^n + \nu\,(u_i^n - u_{i-1}^n) = 0 \quad , \quad \nu = \frac{c\Delta t}{\Delta x}$

Subst. from II, III in I :-

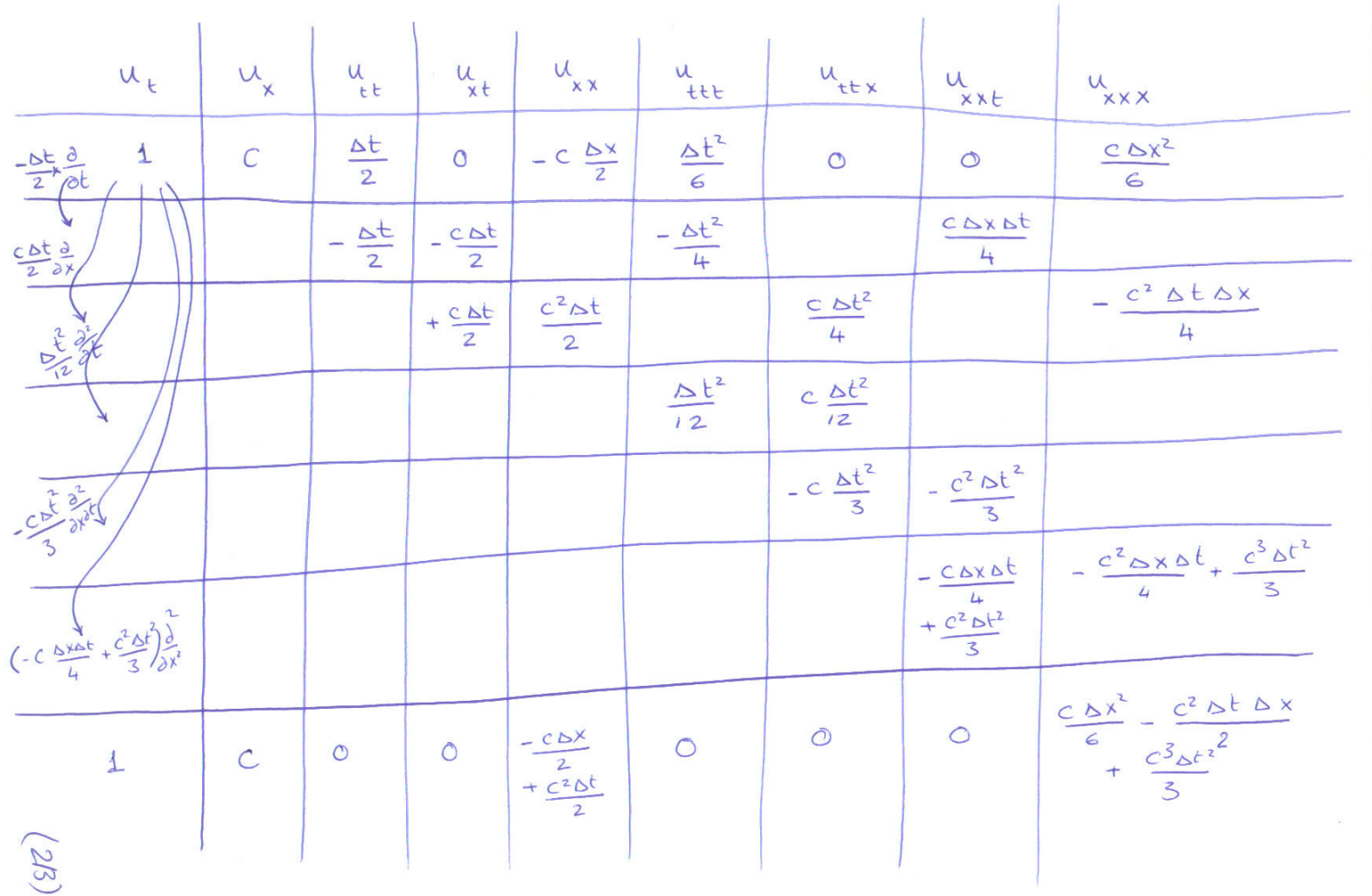
$$\Delta t\, u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} + c\Delta t \left(u_x - \frac{\Delta x}{2} u_{xx} + \frac{\Delta x^2}{6} u_{xxx} \right) = 0$$

$$\Rightarrow \cancel{u_t} + \frac{\Delta t}{2} u_{tt} + \frac{\Delta t^2}{6} u_{ttt} + \cancel{c u_x} - c\frac{\Delta x}{2} u_{xx} + \frac{c\Delta x^2}{6} u_{xxx} = 0$$

$$\Rightarrow u_t + c u_x = -\frac{\Delta t}{2} u_{tt} - \frac{\Delta t^2}{6} u_{ttt} + c\frac{\Delta x}{2} u_{xx} - \frac{c\Delta x^2}{6} u_{xxx}$$

discretization $\Rightarrow$ modified Equation!

... Now try to get rid of t-derivatives.


u_t
u_x
u_{tt}
u_{xt}
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u_{ttt}
u_{ttx}
u_{xxt}
u_{xxx}
1
C
$\frac{\Delta t}{2}$
0
$-c\frac{\Delta x}{2}$
$\frac{\Delta t^2}{6}$
0
0
$\frac{c\Delta x^2}{6}$
$-\frac{\Delta t}{2}\frac{\partial}{\partial t}$
$-\frac{\Delta t}{2}$
$-\frac{c\Delta t}{2}$
$-\frac{\Delta t^2}{4}$
$\frac{c\Delta x\Delta t}{4}$
$\frac{c\Delta t}{2}\frac{\partial}{\partial x}$
$+\frac{c\Delta t}{2}$
$\frac{c^2\Delta t}{2}$
$\frac{c\Delta t^2}{4}$
$-\frac{c^2\Delta t\Delta x}{4}$
$\frac{\Delta t^2}{12}\frac{\partial^2}{\partial t^2}$
$\frac{\Delta t^2}{12}$
$c\frac{\Delta t^2}{12}$
$-\frac{c\Delta t^2}{3}\frac{\partial^2}{\partial x^2 t}$
$-c\frac{\Delta t^2}{3}$
$-\frac{c^2\Delta t^2}{3}$
$\left(-c\frac{\Delta x\Delta t}{4}+\frac{c^2\Delta t^2}{3}\right)\frac{\partial^2}{\partial x^2}$
$-\frac{c\Delta x\Delta t}{4}+\frac{c^2\Delta t^2}{3}$
$-\frac{c^2\Delta x\Delta t}{4}+\frac{c^3\Delta t^2}{3}$
1
C
0
0
$-\frac{c\Delta x}{2}+\frac{c^2\Delta t}{2}$
0
0
0
$\frac{c\Delta x^2}{6}-\frac{c^2\Delta t\Delta x}{3}+\frac{c^3\Delta t^2}{3}$

∴ Modified Equation is :-

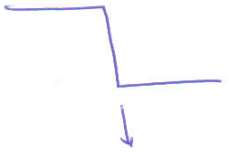
$$u_t + c\,u_x - \underbrace{\frac{c\Delta x}{2}(1-\nu)\,u_{xx}}_{\text{Even deriv.} \rightarrow \text{dissipative Error}} - \frac{c\Delta x^2}{6}(3\nu - 2\nu^2 - 1)\,u_{xxx} = 0$$

Type of Error

Even derivative

u_{xx}

Dissipative Error



"Smoothing"

"Artificial Viscosity"

Affect Accuracy

Odd derivative

u_{xxx}

Dispersive Error

"Oscillations"

"Not Desired"