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Subject: Hydrodynamics "Advanced"

[Signature]
Fluid Mechanics "Review"

Fluids in Motion:

# Velocity: Lagrangian & Eulerian viewpoints

1. The Lagrangian viewpoint:

The position vector for an individual fluid particle at time \( t \) is given by:

\[
\mathbf{r}(t) = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}
\]

and so

\[
\mathbf{V}(t) = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k}
= \mathbf{u} \mathbf{i} + \mathbf{v} \mathbf{j} + \mathbf{w} \mathbf{k}
\]

Hints:

1. Motions of all fluid particles must be considered simultaneously.

2. The motion of the flow field is obtained by solving the equation of motion \( \mathbf{F} = m \ddot{\mathbf{r}} \) for each and every fluid particle.
2. Eulerian Approach:

Focuses on a certain point in space and considers the motion of fluid particles that pass that point as time goes on.

\[ u = f_1(x,y,z,t) \quad \text{and} \quad \omega = f_3(x,y,z,t) \]

\[ v = f_2(x,y,z,t) \]

Hint:

It is an enormous task to keep track of the position of all particles in a flow field because their relative position continuously change with time. So the Eulerian Approach is favored.

\[
\begin{align*}
\dot{x} &= \frac{dx}{dt} = \frac{du}{dt} = d \frac{f_1(x,y,z,t)}{dt} \\
&= \frac{du}{dx} \frac{dx}{dt} + \frac{du}{dy} \frac{dy}{dt} + \frac{du}{dz} \frac{dz}{dt} \\
&= u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}
\end{align*}
\]

And in the same way:

\[
\begin{align*}
\dot{y} &= u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \\
\dot{z} &= u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}
\end{align*}
\]

Convecitive transport depends on position.

Local convective transport depends on time.
Tangential & Normal Acceleration

Another useful way of expressing velocity using position, along stream line & time is given as

\[ V = f(s, t) \]

for \( \hat{i}, \hat{j} \) unit vectors in \( x, y \) directions

\[ \hat{i}_t, \hat{i}_n \] unit vectors in the tangential & normal directions

\[ ds = s \] \[ d\theta = \frac{d\theta}{ds} \]

\[ \hat{i}_t = \cos \theta \hat{i} + \sin \theta \hat{j} \]

\[ \hat{i}_n = -\sin \theta \hat{i} + \cos \theta \hat{j} \]

By definition \( \vec{V} = V \hat{i}_t \)

\[ \vec{a} = \frac{d\vec{V}}{dt} = \frac{dV}{dt} \hat{i}_t + V \frac{d\hat{i}_t}{dt} \]

\[ = \left( \frac{dV}{dt} \right) \hat{i}_t + \frac{d\hat{i}_t}{dt} + V \frac{d\theta}{ds} \frac{ds}{d\theta} \]

\[ = \left( \frac{dV}{dt} \right) \hat{i}_t + \frac{d\hat{i}_n}{dt} + \frac{1}{\rho} \hat{n} \]

\[ = \left( \frac{dV}{ds} \frac{ds}{dt} + \frac{dV}{dt} \right) \hat{i}_t + \frac{V^2}{\rho} \hat{n} \]

\[ = \left( V \frac{d\hat{i}_t}{ds} + \frac{dV}{dt} \right) \hat{i}_t + \frac{V^2}{\rho} \hat{n} \]

convective \hspace{1cm} local
Uniform flow: \( \frac{\partial V}{\partial s} = 0 \)

Steady flow: \( \frac{\partial V}{\partial t} = 0 \)

Turbulent flow: characterized by a mixing action throughout the flow field caused by eddies.

Rate of flow: \( Q = \int_A \vec{V} \cdot d\vec{A} \)

Mass flow rate: \( m = \rho Q = \int_A \rho \vec{V} \cdot d\vec{A} \)

Average or mean velocity: \( \bar{V} = \frac{Q}{A} \)
Angular Velocity, Vorticity, $\Omega_y$:

\[ \Delta y = v \Delta t \]
\[ \Delta y_c = [u + \frac{\partial v}{\partial x} \Delta x] \Delta t \]
\[ \Delta y_c - \Delta y_A = \frac{\partial u}{\partial x} \Delta x \Delta t \]

\[ \tan(\theta_2) = \frac{\frac{\partial u}{\partial x} \Delta x \Delta t}{\frac{\partial v}{\partial y} \Delta y \Delta t} = \frac{\partial u}{\partial x} \Delta t \pm \Delta \theta_2 \]

\[ \frac{\Delta \theta_2}{\Delta t} = \text{rate of rotation} = \frac{\partial v}{\partial x} = \omega_{AC} \]

In the same way:

\[ \Delta x_A = u \Delta t \]
\[ \Delta x_B = [u + \frac{\partial u}{\partial y} + \Delta y] \Delta t \]
\[ \Delta x_B - \Delta x_A = \frac{\partial u}{\partial y} + \Delta y \Delta t \]

\[ \tan(-\theta_1) = \frac{\frac{\partial u}{\partial x} \Delta x \Delta t}{\frac{\partial v}{\partial y} \Delta y \Delta t} = \frac{\partial u}{\partial x} \Delta t \mp \Delta \theta_1 \]

\[ \frac{\Delta \theta_1}{\Delta t} = \frac{\partial u}{\partial y} = \omega_{AB} \]

Define angular velocity:

\[ \omega_{\text{mean}} = \frac{\omega_{AB} + \omega_{AC}}{2} = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \]

\[ \omega_x = \frac{1}{2} \left( \frac{\partial \omega}{\partial y} - \frac{\partial \omega}{\partial z} \right) \]
\[ \omega_y = \frac{1}{2} \left( \frac{\partial \omega}{\partial z} - \frac{\partial \omega}{\partial x} \right) \]
Generally
\[ \vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} \]

\[ = \frac{1}{2} ( \frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial z} ) \hat{i} + \frac{1}{2} ( \frac{\partial v}{\partial z} - \frac{\partial u}{\partial x} ) \hat{j} + \frac{1}{2} ( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} ) \hat{k} \]

Define \( \vec{f} = 2 \vec{\omega} \) "vorticity"

\[ = \left( \frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{i} + \left( \frac{\partial v}{\partial z} - \frac{\partial u}{\partial x} \right) \hat{j} + \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \hat{k} \]

\[ = \nabla \times \vec{V} = \text{curl (velocity vector)} \]

HINTS

IRROTATIONAL FLOW \( \Rightarrow \nabla \times \vec{V} = 0 \) at any point
Basic Control Volume Approach

⇒ Extensive Property: Properties related to the total mass
e.g. mass, energy, momentum

⇒ Intensive Property: Properties that are independent of the
amount of fluid
  e.g. density & pressure

In general
\[ B = \int \beta \cdot dv = \int \beta \cdot \rho \cdot dV \]

⇒ Control Volume: Region in space that one establishes
to aid in the solution of flow problem

Net Flow rate = \[ \sum \frac{\bar{V} \cdot A}{\text{cs}} \]

Also, net mass flow rate
\[ m = \sum \frac{\rho \cdot A \cdot V}{\text{cs}} \text{ or } \sum \rho \cdot dV \cdot \bar{A} \]

\[ B = \sum \frac{\beta \cdot dm}{\text{cs}} = \sum \frac{\beta \cdot \rho \cdot \bar{V} \cdot \bar{A}}{\text{cs}} \]
Derivation of the control volume equations:

\[
\frac{d}{dt} B_{\text{system}} = \lim_{\Delta t \to 0} \frac{B_{\text{end}} - B_0}{\Delta t}
\]

\[
= \lim_{\Delta t \to 0} \frac{(B_2 + B_3)_{t+\Delta t} - (B_1 + B_2)_t}{\Delta t}
\]

\[
= \lim_{\Delta t \to 0} \frac{B_2_{t+\Delta t} - B_2_t}{\Delta t} + \lim_{\Delta t \to 0} \frac{B_3_{t+\Delta t} - B_3_t}{\Delta t}
\]

\[
= \frac{d}{dt} B_{cV} + B_i
\]

\[
= \frac{d}{dt} \int \rho \, dV + \sum_{c} \rho_0 \vec{V} \cdot \vec{A}
\]

rate of change of \(B\) within the control volume

rate of flow of \(B\) out of the control volume

for steady flow \(\int \rho \, dV = 0\)

\[
\Rightarrow \frac{d}{dt} (B_{\text{sys}}) = \sum_{c} \rho_0 \vec{V} \cdot \vec{A}
\]
# Continuity Equations -

Recall \( \frac{d \text{Systems}}{dt} = \frac{d}{dt} \int_{A} \rho \, dv + \sum_{s} \rho_{v} \hat{v} \cdot \hat{A} \)

for \( B = M \), \( \beta = 1 \)

\[ \frac{d}{dt} \text{systems} = \frac{d}{dt} \int_{A} \rho \, dv + \sum_{s} \rho_{v} \hat{v} \cdot \hat{A} = 0 \]

\[ \sum_{s} \frac{\partial}{\partial t} \rho \hat{v} \cdot \hat{A} = - \frac{d}{dt} \int_{A} \rho \, dv \]

Outflow net rate = rate of mass decrease within the C.V

\[ \Rightarrow \text{for steady flow} \]

\[ \sum_{s} \rho \hat{v} \cdot \hat{A} = 0 \Rightarrow - \rho_{1} V_{1} A_{1} + \rho_{2} V_{2} A_{2} = 0 \]

\[ \Rightarrow \text{if incompressible} \Rightarrow V_{1} A_{1} = V_{2} A_{2} \]
Continuity Equation at a point

\[ \sum_{c=3} \rho \vec{v} \cdot \vec{A} = \oint_{\partial c} \rho \vec{v} \cdot d\vec{A} \frac{d\rho}{dt} \cdot \vec{C} \]

Assuming incompressible flow i.e. \( \rho \) constant & steady flow \( \frac{d\rho}{dt} = 0 \)

\[ \frac{\partial}{\partial t} \left( \rho \vec{v} \cdot \vec{A} \right) = 0 \]

\[ \sum_{c=3} \vec{v} \cdot \vec{A} = \frac{\partial u}{\partial x} \ dx \ dy \ dz + \frac{\partial v}{\partial y} \ dx \ dy \ dz \]

\[ + \frac{\partial w}{\partial z} \ dx \ dy \ dz = 0 \]

\[ \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \]

define \( \vec{V} = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \)

so for incompressible steady flow

\[ \vec{V} \cdot \vec{V} = 0 \]
Streamline & Stream Function:
\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \]

Streamline is a curve in space & hence, it can be described by the equation \( f(x, y, z) = 0 \)

- \( \vec{V} \parallel ds \)
- \( \vec{V} \times \vec{s} = 0 \)
- \( ds = dx \hat{i} + dy \hat{j} + dz \hat{k} \)

\[ \left| \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ dx & dy & dz \\ u & v & w \end{array} \right| = 0 \]

\[ (vdx -udy) \hat{i} - (wdx -udy) \hat{j} + (vdx -udy) \hat{k} = 0 \]

\[ vdx = udy \quad \& \quad wdx = udy \]

Now consider 2-D flow:
\[ \frac{dy}{dx} = \frac{v}{u} \]
* Stream function

Given \( u \) & \( v \) as functions in \( x, y \), then

\[
\frac{\partial y}{\partial x} = \frac{v}{u} \quad \Rightarrow \quad \psi(x, y) = c
\]

Define the numerical value of \( \psi \) such that the difference \( \Delta \psi \) is equal to the mass flow between the two streamlines \( \psi \) & \( \psi + \Delta \psi \)

\[
\Delta \psi = \int_0^\Delta n \frac{\partial \psi}{\partial n} \quad \Rightarrow \quad \Delta \psi = \rho u \Delta n
\]

\[
\Rightarrow \quad \frac{\partial \psi}{\partial n} = \rho u \text{ as } \Delta n \to 0 \quad \Rightarrow \quad \frac{\partial \psi}{\partial n} \approx \rho u
\]

Thus we can obtain the product \( \rho u \) by differentiating \( \psi \) in the direction normal to \( \psi \)

\[
\Delta \psi = \int_0^\Delta n \frac{\partial \psi}{\partial n} \quad \Rightarrow \quad \Delta \psi = \rho \int \left[ u \partial x + v \partial y \right]
\]

\[
= \rho \left( u \Delta y - v \Delta x \right)
\]

But \( \Delta \psi = \frac{\partial \psi}{\partial x} \Delta x + \frac{\partial \psi}{\partial y} \Delta y \)

\[
\Rightarrow \quad \Delta \psi = \frac{\partial \psi}{\partial y} \text{ & } \Delta \psi - \frac{\partial \psi}{\partial x}
\]

\[
\Rightarrow \quad u = \frac{\partial \psi}{\partial y} \text{ & } v = -\frac{\partial \psi}{\partial x}
\]

\[
\Rightarrow \quad u = \frac{\partial \psi}{\partial y} \text{ & } v = -\frac{\partial \psi}{\partial x}
\]
# Velocity Potential

An irrotational flow is defined by $$\nabla \times \vec{V} = 0$$

$$\Rightarrow \nabla \times (\nabla \phi) = 0 \quad \& \quad \vec{V} = \nabla \phi$$

i.e. for an irrotational flow there exists a scalar function \( \phi \) such that the velocity is given by the gradient of \( \phi \)

\[
\begin{align*}
\phi & : \quad u \hat{i} + v \hat{j} + w \hat{k} = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \\
\phi & : \quad u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}
\end{align*}
\]

# Relationship between the stream function & velocity potentials

\[
\begin{align*}
d\psi &= \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \quad \Rightarrow \quad u = \frac{\partial \psi}{\partial y} \quad \& \quad v = -\frac{\partial \psi}{\partial x} \\
d\psi &= -v \, dx + u \, dy = 0 \\
\phi \left( \frac{dy}{dx} \right)_{\phi: \text{const}} &= \frac{v}{u}
\end{align*}
\]

Similarly, for an equipotential \( \phi(x, y) \), \( \phi(x, y) = \text{const} \)

\[
\begin{align*}
\phi &= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0 \quad \Rightarrow \quad u = \frac{\partial \phi}{\partial x} \quad \& \quad v = \frac{\partial \phi}{\partial y} \\
d\phi &= u \, dx + v \, dy = 0 \quad \Rightarrow \quad \left( \frac{dy}{dx} \right)_{\phi: \text{const}} = -\frac{u}{v}
\end{align*}
\]
\[
\frac{dy}{dx}_{y=\text{const}} \times \frac{dy}{dx}_{\phi=\text{const}} = -1
\]

i.e. streamlines & equipotential lines are mutually perpendicular.