Unsteady Incompressible Flow Simulation using Galerkin Finite Elements with Spatial/Temporal Adaptation

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47th AIAA Aerospace Sciences Meeting

January 07, 2009
Overview

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2 Algorithm

3 Results
   NACA Airfoil
   2-Vertical Cylinders

4 Recent Development and Future Work
   Turbulence Modeling
   All-quad meshing with guaranteed quality
Motivation

Main Targets:

- Galerkin Finite Element discretization
  - Zero Mach Number.
  - Primitive variables \((u, v, p)\).
  - Stable (Mixed approximation).
- Fast Iterative methods.
  - Krylov subspace methods (GMRES, PCG, TFQMR)
  - Multigrid methods.
- A dynamic remeshing technique.
  - Fast and adaptive.
  - No hanging nodes.
  - Guaranteed quality.
  - Easy implementation of multigrid methods.
  - Suitable for line solvers.
- Adaptive selection for the time step.
Algorithm: Governing Equations

Unsteady Incompressible Navier Stokes Eq

\[
\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla \mathbf{q} - \frac{1}{Re} \nabla^2 \mathbf{q} + \nabla p = \mathbf{f}
\]

\[\nabla \cdot \mathbf{q} = 0\]  

Boundary Conditions

\[
\mathbf{q} = \mathbf{w} \text{ on } \partial \Omega_D, \quad \frac{1}{Re} \frac{\partial \mathbf{q}}{\partial n} - \mathbf{n} p = 0 \text{ on } \partial \Omega_N
\]
Algorithm: Weak Formulation

Standard Weak Formulation

If we define the solution and test spaces

\[ H^1_E := \{ u \in H^1(\Omega)^d \mid u = w \text{ on } \partial \Omega_d \} \]
\[ H^1_{E_0} := \{ v \in H^1(\Omega)^d \mid v = 0 \text{ on } \partial \Omega_d \} \]

then the standard weak formulation is to find \( u \in H^1_E \) and \( p \in L^2(\Omega) \) such that

\[
\int_{\Omega} \frac{\partial u}{\partial t} \cdot v + \int_{\Omega} (u \cdot \nabla u) \cdot v + \frac{1}{Re} \int_{\Omega} \nabla u : \nabla v + \int_{\Omega} p(\nabla \cdot v) = \int_{\Omega} f \cdot v , \quad \forall \ v \in H^1_{E_0}
\]

\[
\int_{\Omega} q(\nabla \cdot u) = 0 , \quad \forall \ q \in L^2(\Omega)
\]
Mixed Finite Element Approximation

A discrete weak formulation is defined using finite dimensional spaces $X^h_o \subset H^1_{E_o}$ and $M^h \subset L^2(\Omega)$. The discrete problem is to find $u_h \in X^h_E$ and $p_h \in M^h$ such that

$$
\int_\Omega \frac{\partial u}{\partial t} \cdot v + \int_\Omega (u \cdot \nabla u) \cdot v + \frac{1}{Re} \int_\Omega \nabla u : \nabla v + \int_\Omega p(\nabla \cdot v) = \int_\Omega f \cdot v , \ \forall \ v \in X^h_o
$$

$$
\int_\Omega q(\nabla \cdot u) = 0 , \ \forall \ q \in M^h(\Omega)
$$
Algorithm: Discretization

Nonlinear system

\[ q_{ij} \left( u^n_j - u^{n-1}_j \right) + c_{ijk} u^n_j u^n_k + a_{ij} u^n_j + b_{lj} q_l = s_{ij} f_j \]

\[ b_{lj} u^n_j = 0 \]

Here, \( n \) represents time, \( i, j = 1, 2, \ldots, n_u, l = 1, 2, \ldots, n_p \). The operators \( q_{ij}, a_{ij}, b_{ij}, s_{ij} \) represents sparse matrices defined as follows:

\[ q_{ij} \in \mathbb{R}^{n_u \times n_u} = \text{Mass Matrix} = \frac{1}{\Delta t} \int_\Omega \phi_j \cdot \phi_i \]

\[ a_{ij} \in \mathbb{R}^{n_u \times n_u} = \text{Diffusion Operator} = \int_\Omega \nabla \phi_j : \nabla \phi_i \]

\[ b_{lj} \in \mathbb{R}^{n_p \times n_u} = \text{Divergence Operator} = \int_\Omega \psi_l (\nabla \cdot \phi_j) \]

while the operator \( c_{ijk} \) is a sparse 3d tensor defined as follows

\[ c_{ijk} \in \mathbb{R}^{n_u \times n_u \times n_u} = \text{Convection Operator} = \int_\Omega (\phi_k \cdot \nabla \phi_j) \cdot \phi_i \]
Picard’s Iteration

- Starts with an ‘initial guess’ \((u^{n,o}, p^{n,o})\).
- A sequence of iterates \(\{(u^{n,m}, p^{n,m})\}\) is constructed till it converges to the solution of the weak formulation.
- Nonlinear convection operator:

\[
c(u^n, u^n, v) \approx c(u^{n,m}, u^{n,m+1}, v)
\]

so we can now define the operator

\[
c_{ij}^m = c_{ijk} u_{k}^{n,m}
\]
Algorithm: Solution of the linearized system

The linear system associated with incompressible flows

\[
\begin{bmatrix}
  F & B^T \\
  B & 0
\end{bmatrix}
\begin{bmatrix}
  u \\
p
\end{bmatrix}
= 
\begin{bmatrix}
  f^* \\
g
\end{bmatrix}
\]

(4)

- Indefinite system so it needs special iterative techniques (GMRES) to converge.
- However, we need an efficient preconditioner to ensure a reasonable convergence.
Algorithm: Preconditioned Linear system

The Least Square Commutator

The Shur complement matrix is approximated as follows

\[ BF^{-1}B^T \approx (BQ^{-1}B^T)(BQ^{-1}FQ^{-1}B^T)^{-1}(BQ^{-1}B^T) \]

Now the right preconditioned linear problem is given by

\[
\begin{bmatrix}
F & B^T \\
B & 0
\end{bmatrix}
\begin{bmatrix}
F & B^T \\
0 & -M_S
\end{bmatrix}^{-1}
\begin{bmatrix}
u^* \\
p^*
\end{bmatrix}
= \begin{bmatrix}
f \\
g
\end{bmatrix}
\]

(5)

where

\[
\begin{bmatrix}
u^* \\
p^*
\end{bmatrix}
= \begin{bmatrix}
F & B^T \\
0 & -M_S
\end{bmatrix}
\begin{bmatrix}
u \\
p
\end{bmatrix}
\]
Algorithm: Dynamic Remeshing

Motivation

Algorithm

Results

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2-Vertical Cylinders

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Turbulence Modeling All-quad
Algorithm: Time Step Control

Choosing Time Step based on error estimation

we have the following error estimation

\[ u \left( t; \frac{h}{2} \right) - u^*(t) \approx \frac{u(t; h) - u \left( t; \frac{h}{2} \right)}{2^p - 1} \]

At each time step we start with an initial guess \( H > 0 \) for the time step, calculate

\[ r_H = \frac{H}{h} \approx \left( \frac{2^p}{2^p - 1} \left\| u(t_o + H; H) - u \left( t_o + H; \frac{H}{2} \right) \right\| \frac{1}{\epsilon} \right)^{\frac{1}{p + 1}} \]
Algorithm
Timestep Evolution: Impulsive flow over a cylinder at Re = 1200
Results: Unsteady Flow over a NACA0012 Airfoil
Re=800, $\alpha = 20^\circ$

Run Movie 1
Results: Unsteady Flow over a NACA0012 Airfoil
Re=800, \( \alpha = 20^\circ \)

Run Movie 2
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Results: Unsteady Flow over 2 Vertical Cylinders
Re=200

Run Movie 3
Results: Unsteady Flow over 2 Vertical Cylinders: Multigrid Levels

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Results: Unsteady Flow over 2 Vertical Cylinders: Multigrid Levels
Turbulence Modeling: Boundary Layer
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Turbulence Modeling: Multi Element Airfoil

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Turbulent Flow Over a Flat plate $Re=10^5$

Grid

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Velocity Profile

- Viscous Sublayer
- Logarithmic Law
- Computed

$y^+ \text{ Log Scale}$

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Turbulent Flow Over a Flat plate $Re=10^5$

Shear Stress

![Graph of shear stress over a flat plate with $Re=10^5$.](image)
Handling of Complex Geometries

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Future Work

- Application of Multigrid methods.
- Turbulence modeling.
- Moving boundaries.
- Extension to 3D.

• Application of Multigrid methods.
• Turbulence modeling.
• Moving boundaries.
• Extension to 3D.
Thank you!